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Space, Time, and Velocity in Cosmology

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In the limit of negligible gravity, a transformation that relates physical quantities at different cosmic times, similar to the Lorentz transformation which relates measurements at different velocities, is derived.

1. INTRODUCTION

In prerelativistic physics it was assumed that space is not related to time; a "stationary" frame of reference was presumed to exist with respect to which all physical phenomena can be described.

As Einstein (1905) showed, this picture was wrong; space has no preference of a particular frame or any other one that moves with a constant velocity, and in this way one can accommodate the fact that light propagates with a constant velocity in all moving systems. The mixture of space and time became a necessity in order to preserve the constancy of the propagation of light in all inertial frames. The mathematical expression of this fact is given by the familiar Lorentz transformation, which was rederived by Einstein, who also gave to it the correct physical interpretation.

As Bernard Russell said, "Einstein's theory of relativity is probably the greatest synthetic achievement of the human intellect up to the present time."

2. FUNDAMENTALS OF SPECIAL RELATIVITY

The essence of the theory of special relativity, according to Einstein (1979), is as follows:

> According to the rules of connection, used in classical physics, between the spatial coordinates and the time of events in the transition from one inertial system to another, the two assumptions of

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(1) the constancy of the light velocity

(2) the independence of the laws (thus especially also of the law of the constancy of the light velocity) from the choice of inertial system

are mutually incompatible (despite the fact that both taken separately are based on experience).

The insight fundamental for the special theory of relativity is this: The assumptions (1) and (2) are compatible if relations of a new type (Lorentz transformation) are postulated for the conversion of coordinates and times of events. With the given physical interpretation of coordinates and time, this is by no means merely a conventional step but implies certain hypotheses concerning the actual behavior of moving measuring rods and clocks, which can be experimentally confirmed or disproved.

The universal principle of the special theory of relativity is contained in the postulates: The laws of physics are invariant with respect to Lorentz transformations (for the transition from one inertial system to any other arbitrarily chosen inertial system). This is a restricting principle for natural laws, comparable to the restricting principle of the nonexistence of the *perpetuum mobile* that underlies thermodynamics.

3. PRESENT-DAY COSMOLOGY

At present we have a similar situation in cosmology to that existing in the prerelativistic era with respect to space and (not velocity, but) cosmic time, in conjunction with the constancy of expansion of the universe (and not propagation of light). If we take the convention according to which cosmic time, denoted by t, is measured backward, then our present time ($t = 0$) is a preferred time with respect to which all cosmological physical phenomena are referred. This is exactly analogous to the prerelativity assumption that physical phenomena referred to only one "stationary" ($v = 0$) system.

Actually, space has no such preference: When we consider an astronomical object and say that it is, say, at $t = \tau/2$, where $\tau = 1/H_0$ is Hubble's time, that faraway object has the same right to say that it is at cosmic time zero ($t = 0$) and we are at $t = \tau/2$ with respect to it, exactly as in relativistic physics, but with the roles of cosmic time and velocity exchanged. We will assume that such a reciprocity relationship between cosmological objects is a universal property of space and cosmic time, just as Einstein did with respect to space and velocity in special relativity.

4. POSTULATES

In addition, we will make two assumptions which will be elevated to postulates. These are: (1) *The principle of the constancy of the expansion of the universe* (expressed by Hubble's law) at all cosmic times (analogous to the principle of the constancy of propagation of light in all moving frames);

and (2) *the principle of cosmological relativity* (analogous to the principle of special relativity) according to which the laws of physics are the same at all cosmic times (at moving frames in special relativity).

5. COSMIC FRAMES

In this way the universe has *cosmic frames of reference* located at fixed cosmic times and differing from each other by relative constant cosmic times, similar to the situation in special relativity, but now with cosmic times replacing velocities. Observers in each cosmic frame are equipped with a ruler to measure distance (as in special relativity) and with a small radar device (similar to that used by the highway patrol) for velocity measurements (instead of clocks in special relativity).

Notice the analogy between the relation $[\tau]$ = distance/velocity in the present theory and $[c]$ = distance/time in special relativity, which suggests the choice of distance and velocity as our fundamental variables as compared to distance and time in special relativity.

Remark. The constant τ is used by us just as the constant c is used in special relativity even though it is well known that both the speed of light and the rate of expansion of the universe change their values due to gravity. This is possible since local measurements of both the velocity of light and the rate of expansion of the universe always yield constant c and τ, respectively.

6. SPACE AND VELOCITY IN COSMOLOGY

With the above postulates, and by comparison with special relativity, it is obvious that space and velocity cannot be independent if Hubble's law is to be preserved at all cosmic times. In fact this will enable us to derive a transformation that relates space points and velocities (and other quantities) measured in different cosmic frames of reference that differ in relative cosmic times, just like the Lorentz transformation, which relates space points and time (and other quantities) measured in different inertial frames that differ in relative velocities. Space coordinates and velocities become unified in cosmology just as space and time are unified in local (noncosmological) physics.

7. PRESPECIAL RELATIVITY

With the above preliminaries we are now in a position to develop our theory. To begin with, we repeat very briefly what preceded special relativity.

The Galilean transformation between two inertial systems K and K', where K' moves relative to K with a constant velocity v along the x axis, is given by

$$x' = x - vt, \qquad t' = t, \qquad y' = y, \qquad z' = z$$

Here x and x' represent the coordinates of a particle in the systems K and K', respectively.

The trouble with the Galilean transformation is its incompatibility with the equation of propagation of light, which satisfies

$$c^2t'^2 - x'^2 = c^2t^2 - x^2, \qquad y' = y, \qquad z' = z$$

Hence the Galilean transformation should be replaced by a new one that relates not only x' to x leaving t unchanged, but relates x' *and* t' to x *and* t. And this immediately leads to the familiar Lorentz transformation.

8. RELATIVE COSMIC TIME

In cosmology one is not interested in comparing quantities at two reference frames moving with a constant velocity with respect to each other. Rather, one is interested in comparing quantities in two different cosmic times. For example, one often asks what was the density of matter or the temperature of the universe at an earlier time t as compared to the values of these quantities at our present time now ($t = 0$). The backward time t is the *relative* time with respect to our present time.

The concept of the relative time is not restricted only to the backward time t with respect to the present time ($t = 0$). Every two observers with times t_1 and t_2 with respect to us are related to each other by a relative time t. Thus t plays the role of the velocity v in special relativity and we will see in the sequel that t has an upper limit, which is the Hubble time τ, just as the maximum velocity permitted in special relativity is c.

The variables (coordinates) in this theory are naturally the Hubble variables, i.e., the velocity v and the distance x. To derive the transformation between these variables in the systems K and K', where K' has a relative time t with respect to K, we proceed as follows.

9. INADEQUACY OF CLASSICAL TRANSFORMATION

We first do this classically, and for simplicity it is assumed that the motion is one-dimensional. Denoting the coordinates and velocities in the systems K and K' by x, v and x', v', respectively, then

$$x' = x - tv, \qquad v' = v, \qquad y' = y, \qquad z' = z$$

where v is assumed to be constant. The x's and v's in these equations represent the coordinates and velocities not for just one particle, but for as many as one wishes, with t the same for all of them.

The above transformation does not satisfy the equation of expansion of the universe, which, according to the principle of the constancy of expansion of the universe and the principle of cosmological relativity demanding the laws of physics (and in particular Hubble's law) to be valid at all cosmic times, satisfies

$$\tau^2 v'^2 - x'^2 = \tau^2 v^2 - x^2, \qquad y' = y, \qquad z' = z$$

The situation here is similar to what we had at the beginning of the century, where the Galilean transformation could not accommodate both the principle of special relativity and the principle of the constancy of the speed of light, leading to the Lorentz transformation. A new transformation here also has to be found which relates not only x' to x leaving v unchanged, but relates x' *and* v' to x *and* v.

10. UNIVERSE EXPANSION VERSUS LIGHT PROPAGATION

Under the assumption that Hubble's constant is constant in cosmic time, there is an analogy between the propagation of light, $x = ct$, and the expansion of the universe, $x = \tau v$, where τ is the Hubble time, a constant which is also the age of the universe under the above assumption, and c is the speed of light in vacuum. Thus one can express the expansion of the universe, assuming that it is homogeneous and isotropic, in terms of a null vector satisfying

$$x^2 + y^2 + z^2 - \tau^2 v^2 = 0 \tag{1}$$

where v is the receding velocity of the galaxies. Equation (1), in the four-dimensional flat space of the Cartesian three-space and velocity, is similar to

$$x^2 + y^2 + z^2 - c^2 t^2 = 0 \tag{2}$$

for the null propagation of light in Minkowskian spacetime. We assume, furthermore, that a relationship of the form (1) is valid at all cosmic times. Thus, at a cosmic time t' at which the coordinates and velocity are labeled with primes, we have

$$x'^2 + y'^2 + z'^2 - \tau^2 v'^2 = 0 \tag{3}$$

with the same τ, just as for light emitted from a source with velocity v with respect to the first one,

$$x'^2 + y'^2 + z'^2 - c^2 t'^2 = 0. \tag{4}$$

Accordingly, we have a four-dimensional space with zero curvature of x, y, z, v just like the Minkowskian spacetime of x, y, z, t.

We now assume that at two cosmic times t and t' we have

$$x'^2 + y'^2 + z'^2 - \tau^2 v'^2 = x^2 + y^2 + z^2 - \tau^2 v^2 \tag{5}$$

in analogy to the special-relativistic formula

$$x'^2 + y'^2 + z'^2 - c^2 t' = x^2 + y^2 + z^2 - c^2 t^2 \tag{6}$$

The question is then, what is the transformation between x', y', z', v' and x, y, z, v that satisfies the invariance formula (5)?

11. DERIVATION OF THE TRANSFORMATION

For simplicity we again assume that the motion is along the x axis. Hence Hubble's law in the systems K and K' is given by

$$x = \tau v, \qquad x' = \tau v' \tag{7}$$

where x, v and x', v' are measured in K and K', respectively. Assuming now that x, v and x', v' transform linearly, then

$$x' = ax - bv \tag{8}$$

$$x = ax' + bv' \tag{9}$$

where a and b are some variables which are independent of the coordinates.

At $x' = 0$ and $x = 0$, equations (8) and (9) yield, respectively,

$$\frac{b}{a} = \frac{x}{v} = t \tag{10}$$

and

$$\frac{b}{a} = -\frac{x'}{v'} = t \tag{11}$$

Using now equations (7)–(9), we obtain

$$\tau v = x = ax' + bv' = a\tau v' + bv' = (a\tau + b)v' \tag{12a}$$

and similarly

$$\tau v' = (a\tau - b)v \tag{12b}$$

Eliminating v and v' from equations (12a) and (12b) and using $b = at$ from equation (10), we get

$$\tau^2 = a^2(\tau^2 - t^2) \tag{13}$$

or

$$a = 1/(1 - t^2/\tau^2)^{1/2} \tag{14}$$

and therefore

$$b = t/(1 - t^2/\tau^2)^{1/2} \tag{15}$$

Inserting these results in equations (8) and (9), we obtain

$$x' = \frac{x - tv}{(1 - t^2/\tau^2)^{1/2}}, \qquad v' = \frac{v - xt/\tau^2}{(1 - t^2/\tau^2)^{1/2}} \tag{16}$$

$$y' = y, \qquad z' = z$$

and

$$x = \frac{x' + tv'}{(1 - t^2/\tau^2)^{1/2}}, \qquad v = \frac{v' + x't/\tau^2}{(1 - t^2/\tau^2)^{1/2}} \tag{17}$$

$$y = y', \qquad z = z'$$

for the inverse transformation.

12. INTERPRETATION OF THE TRANSFORMATION

Equations (16) give the transformed values of x and v as measured in the system K' with a relative time t with respect to K. The roles of the time and the velocity are *exchanged* as compared to special relativity. This fits our needs in cosmology, where one measures distances and velocities at different times in the past. The parameter t/τ replaces v/c of special relativity.

It should be emphasized that the transformation (16) is not a trivial exchange of v/c, appearing in the Lorentz transformation, and t/τ here. For example, the red shift $z = v/c$ at low velocities, but is certainly not equal to t/τ for small t/τ. (Details of the red shift are given below.)

13. ANOTHER DERIVATION

The transformation (16) could also have been derived as in the standard derivation of the Lorentz transformation by writing

$$x'^2 - \tau^2 v'^2 = x^2 - \tau^2 v^2 \tag{18}$$

whose solution is

$$x' = x \cosh \psi - v\tau \sinh \psi \tag{19}$$

$$v' = v \cosh \psi - (x/\tau)\sinh \psi$$

At $x' = 0$ we obtain

$$\tanh \psi = x/\tau v = t/\tau \tag{20}$$

and therefore

$$\sinh \psi = \frac{t/\tau}{(1 - t^2/\tau^2)^{1/2}} \tag{21a}$$

$$\cosh \psi = \frac{1}{(1 - t^2/\tau^2)^{1/2}} \tag{21b}$$

which lead to the transformation (16).

The geometrical description of the galaxy cone in the present theory and its comparable light cone in special relativity are given in Figs. 1 and

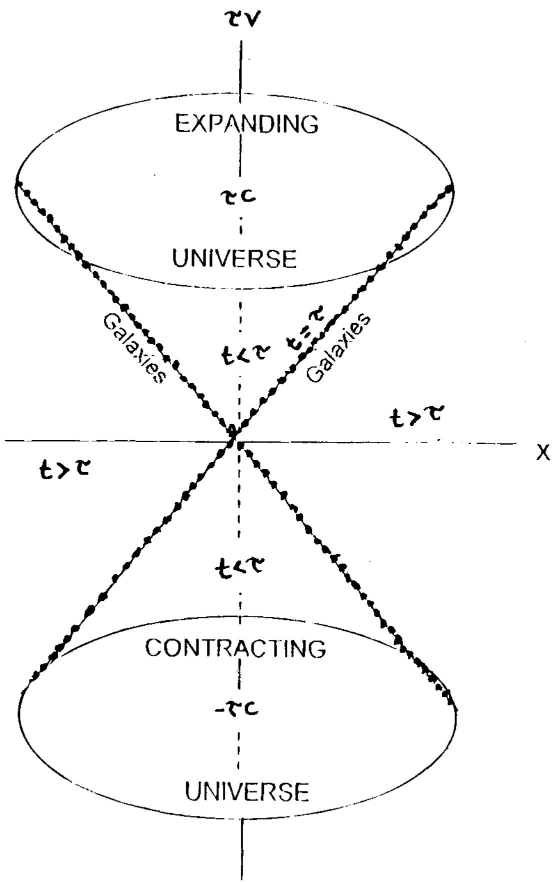

Fig. 1. The galaxy cone in the x–v space satisfying $x^2 - \tau^2 v^2 = 0$ in cosmological special relativity. The dots represent galaxies. The cone represents the location of the galaxies rather than their path of motion.

2. Note that while the light cone expresses the actual motion of light in spacetime, the galaxy cone describes the accumulation or distribution of galaxies in the space-velocity.

14. CONSEQUENCES OF THE TRANSFORMATION

In the following we draw some consequences of the transformations (16) and (17).

14.1. Classical Limit

Assuming that t is much smaller than τ, one can neglect t^2 with respect to τ^2, and the transformation (16) gives

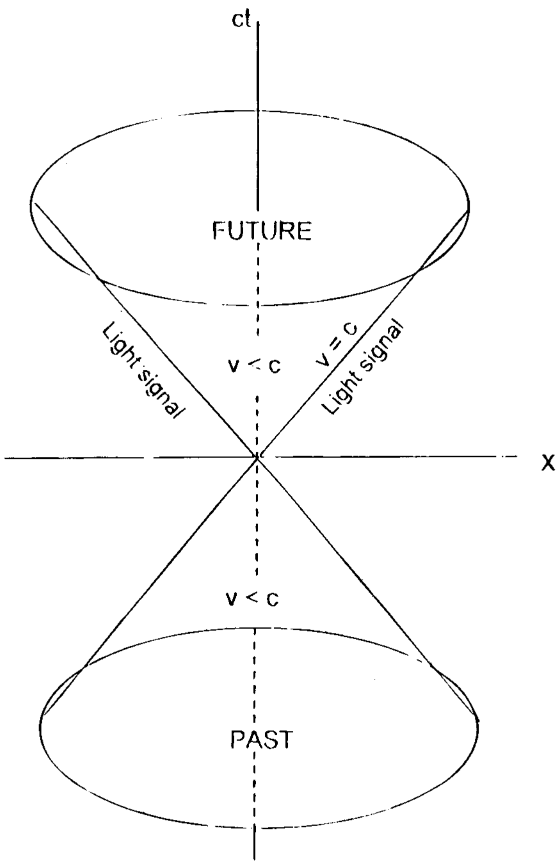

Fig. 2. The light cone in the x–t space of ordinary special relativity.

$$x' = x - tv, \qquad v' = v, \qquad y' = y, \qquad z' = z \tag{22}$$

which is exactly the transformation obtained from classical mechanics.

14.2. Length Contraction

Suppose there is a rod located in the K system parallel to the x axis. Let its length measured in this system be $\Delta x = x_2 - x_1$, where x_2 and x_1 are the coordinates of the two ends of the rod. To determine the length of this rod as measured in the K' system we must find the coordinates of the two ends of the rod x_1' and x_2' in this system at the same velocity v'. From (17) we have

$$x_1 = \frac{x_1' + tv'}{(1 - t^2/\tau^2)^{1/2}}, \qquad x_2 = \frac{x_2' + tv'}{(1 - t^2/\tau^2)^{1/2}}$$

The length of the rod in the K' system is $\Delta x' = x_2' - x_1'$; thus

$$\Delta x = \frac{\Delta x'}{(1 - t^2/\tau^2)^{1/2}}$$

The proper length of a rod is its length in the system in which it is located. Let us denote it by $L_0 = \Delta x$ and the length of the rod in any other system K' by L. Then

$$L = L_0(1 - t^2/\tau^2)^{1/2} \tag{23}$$

Thus a rod has its greatest length in the system in which its relative time with respect to the system is zero; its length in a system in which it is located at a relative time t with respect to that system is decreased by the factor $(1 - t^2/\tau^2)^{1/2}$. This result of the present theory is exactly similar to the familiar Lorentz contraction with the factor $(1 - v^2/c^2)^{1/2}$ in special relativity.

14.3. Velocity Contraction

Suppose a velocity-measuring instrument is located at $x' = 0$ in the K' system. Then from (17) we have

$$v = \frac{v'}{(1 - t^2/\tau^2)^{1/2}} \tag{24}$$

Denote now v by v_0 and v' by v; we obtain

$$v = v_0(1 - t^2/\tau^2)^{1/2} \tag{25}$$

The above result is like the time dilation in special relativity and was expected, since time in special relativity goes over to velocity in the present theory. The velocity measured by an observer with a relative time t with respect to us is smaller by the factor $(1 - t^2/\tau^2)^{1/2}$ than what is observed by us at $t = 0$.

Remark on Dark Matter. As is well known, much of the support for the existence of dark matter is due to the observed very high velocities of gas molecules or galaxies. For example, galaxies in the far-off Coma cluster are observed whirling around one another faster than the laws of physics would allow. So is the mysteriously rapid rotation of spiral galaxies. Equation (25) clearly shows that the velocity observed by us is not the velocity measured by a local observer at a relative time t with respect to us. This observer measures a smaller velocity, and the further back in time, the more the velocity decreases. Does this mean that the hypothetical dark matter can be abolished just as the "luminiferous ether" was proved to be superfluous by special relativity? We will see in a coming paper that this is not so.

14.4. Addition of Times

Dividing the first of (17) by the second, we find, choosing $t = t_1$,

$$\frac{x}{v} = \frac{x' + t_1 v'}{v' + (t_1/\tau^2)x'} \tag{26}$$

or, dividing the numerator and the denominator of the right-hand side of this equation by v', we obtain

$$t = \frac{t_1 + t_2}{1 + t_1 t_2/\tau^2} \tag{27}$$

where $t_2 = x'/v'$ and $t = x/v$.

Equation (27) determines the transformation of time and describes the law of composition of times in cosmological relativity. In the limiting case of t much smaller than τ, equation (27) goes over into the formula $t = t_1 + t_2$ of classical physics.

We see that the simple law of adding and subtracting cosmic times is no longer valid, or, more precisely, is only approximately valid for short times with respect to us, but not for those near the Hubble time, which is also the age of the universe in this case. Consider two consecutive events that occur at $t_1 = (9/10)\tau$ and $t = (180/181)\tau$ both with respect to us (at $t = 0$), for example, with respect to the first event the second one does not occur at $t - t_1 \sim \tau/10$, but rather at

$$t_2 = \frac{t - t_1}{1 - tt_1/\tau^2} = \frac{9}{10}\tau$$

which is much longer than $t - t_1$ and happens to be exactly equal to t_1. We also notice that the past cosmic time cannot be greater than τ, the age of the universe. This is similar to what we have in special relativity, where the velocity cannot exceed c. It will be noted that one may add as many subsequent time intervals as one wishes without ever reaching the age of the universe τ.

14.5. The Line Element

This is given by

$$\tau^2\, dv^2 - (dx^2 + dy^2 + dz^2) = ds^2 \tag{28}$$

Hence

$$\tau^2\left(\frac{dv}{ds}\right)^2 - \left[\left(\frac{dx}{dv}\right)^2 + \left(\frac{dy}{dv}\right)^2 + \left(\frac{dz}{dv}\right)^2\right]\left(\frac{dv}{ds}\right)^2 = (\tau^2 - t^2)\left(\frac{dv}{ds}\right)^2 = 1 \tag{29}$$

Multiplying this equation by ρ_0^2, the matter density of the universe at the present time, we obtain for the matter density at a past time t

$$\rho = \tau\rho_0 \frac{dv}{ds} = \frac{\rho_0}{(1 - t^2/\tau^2)^{1/2}} \tag{30}$$

Remark. Since the volume of the universe is inversely proportional to its density, it follows that the ratio of the volumes at two backward times t_1 and t_2 with respect to us is given by ($t_2 < t_1$)

$$\frac{V_2}{V_1} = \left(\frac{1 - t_2^2/\tau^2}{1 - t_1^2/\tau^2}\right)^{1/2} = \left(\frac{(\tau - t_2)(\tau + t_2)}{(\tau - t_1)(\tau + t_1)}\right)^{1/2}$$

For times t_1 and t_2 very close to τ we can assume that $\tau + t_2 \sim \tau + t_1 \sim 2\tau$. Hence

$$\frac{V_2}{V_1} = \left(\frac{T_2}{T_1}\right)^{1/2}$$

where $T_1 = \tau - t_1$ and $T_2 = \tau - t_2$. For $T_2 - T_1 \approx 10^{-32}$ sec and $T_2 <<$ 1 sec, we then have

$$V_2/V_1 \approx (1 + 10^{-32}/T_1)^{1/2} \approx (10^{-32}/T_1)^{1/2} = 10^{-16}/T_1^{1/2}$$

For $T_1 \sim 10^{-132}$ sec we obtain $V_2 \sim 10^{50} V_1$. This result agrees with an inflationary universe without assuming any specific model (such as that the universe is propelled by a sort of antigravity) (Guth, 1981; Linde, 1982).

14.6. Minimal Acceleration in Nature

From equation (10) we have

$$t = x/v = dx/dv = v/a$$

where a is the acceleration. Hence

$$t_{\max} = \tau = (v/a)_{\max} = c/a_{\min}$$

In nature there is a minimal acceleration

$$a_{\min} = c\tau^{-1} \sim 10^{-8}\ \mathrm{cm/sec^2}$$

Note that such a minimal acceleration constant was proposed previously, but without an explanation for its origin (Milgrom, 1983).

14.7. Cosmological Red Shift

The wavelength of light is inversely proportional to the interval of length as measured by two observers at different cosmic times. The result is $\lambda/\lambda_0 = (1 - t^2/\tau^2)^{-1/2}$. Thus the wavelength of light emitted from a source back in time increases as compared to its value as observed on earth. For $t/\tau << 1$, we have $z = \lambda/\lambda_0 - 1 \sim t^2/2\tau^2$.

15. CONCLUDING REMARKS

The above cosmological special relativity corresponds to a universe with zero curvature, i.e., $\Omega = \rho/\rho_c = 1$; thus $\rho = \rho_c = 3/8\pi G\tau^2 \approx 10^{-29}\ \mathrm{g/cm^3}$, a few hydrogen atoms per cubic meter, is the vacuum energy density, and ρ is the mean mass density. Due to the flatness of the space-velocity in this particular case, and only in this case (other cases are $\Omega > 1$ and $\Omega < 1$), a cosmological special relativity could be developed, since in the $\Omega > 1$ and $\Omega < 1$ cases the space-velocity is not flat. In a sense the theory presented here is half-dynamical, since $\rho \neq 0$, as opposed to ordinary special relativity, which in a sense can be considered as kinematical. It is for this reason that we could obtain results similar to those obtained from the inflationary universe model.

It appears, furthermore, that space, time, and the Hubble expansion of the universe can be unified into one group of five-dimensional transformations that leave invariant the quadratic form

$$c^2t^2 - (x^2 + y^2 + z^2) + \tau^2v^2 = \mathrm{Inv}$$

When the velocity v is constant the reference frame is inertial and we are left with the homogeneous Lorentz group. At a fixed instant of time, on the other hand, we are left with the cosmological group and the reference frame is cosmic. [For more details on this unification see Carmeli (1995).]

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